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The structure of the most extended states of the two-dimensional non-interacting electron gas in a normal magnetic field

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Abstract. The inverse participation ratio and fractal dimension are calculated for states of a tight binding model of a two-dimensional non-interacting electron gas in a normal magnetic field with a random potential. Interest is concentrated on the most extended states. The results are compared with the literature on the fractal nature of the extended states of the continuum version of the model. Substantial differences are found between the behaviour of the fractal dimension of the models. The relevance of the findings to the quantum Hall effect is discussed. It is concluded that arguments linking the behaviour of the fractal dimension of the most extended state to its supposed capacity to carry all current for the subband in the continuum model are not essential for understanding the quantum Hall effect.

The inverse participation ratio (IPR) has been used extensively as a measure of the degree of localisation of electron states caused by a random potential, beginning with the work of Bell and Dean (1970). Since then it has been used as a localisation criterion in both numerical and analytical calculations. For infinite systems it is a strong localisation criterion. It has been shown by Kunz and Soulliard (1980) and Johnston (1981) that the IPR is non-zero only if there is no continuous part to the spectrum of the Hamiltonian operator in the energy range under consideration.

In this paper the IPR will be calculated numerically for finite systems. The Hamiltonian has only a discrete spectrum and so the existence of extended states can only be surmised by extrapolation. The inverse participation for a state is defined in a finite system as

$$P(E_\alpha) = \sum_{r\in\Lambda} |\psi_\alpha(r)|^4 \tag{1}$$

for a d-dimensional lattice Λ and as

$$P(E_\alpha) = \int_\Omega |\psi_\alpha(r)|^4 \, \mathrm{d}^d r \tag{2}$$

for a finite continuous subset Ω of the d-dimensional Euclidean space. ψ_α is the αth normalised eigenstate and E_α is the αth eigenvalue of the Hamiltonian defined on Λ or Ω, with the appropriate boundary conditions.

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It has been noticed that there is a correspondence between $P(E_\alpha)$ and the second-order Renyi entropy (see Paladin and Vulpiani 1987). From this it is evident that the natural fractal dimension associated with the IPR is the second order Renyi dimension

$$D_2 = -\lim_{L\to\infty} \frac{\ln(P(E_\alpha))}{\ln(L)} \tag{3}$$

where L is the length of Λ or Ω. Notice that this dimension is equivalent to other fractal dimensions, which may be defined, if the fractal structure associated with the state is homogeneous.

The specific problem which will be addressed in this paper is the nature of the states of the two-dimensional non-interacting electron gas moving in a random potential with a magnetic field applied normal to the surface. This is a model which has often been invoked to explain the quantum Hall effect (QHE). Two versions are usually considered: one where the configuration space is a continuum and the other where it is a lattice.

Kramer *et al* (1988) and Ono *et al* (1989) have calculated D_2 (their d^*) for a state at the centre of the first Landau band of the continuum model. Aoki (1986) has calculated the Hausdorff dimension for the same state in the same model although he does not discuss the effect of the *ad hoc* cut-off parameter he uses to construct his fractal set.

The point of major interest here is the possibility that the increase in dimensionality of the state at the centre of the band, as the disorder is increased, explains how this state carries extra current and so compensates for the states which become localised. Universality is invoked to imply that the effect should be independent of the microscopic details of the system (Kramer *et al* 1989).

It is well known that the tight binding model can also account for the properties of a two-dimensional electron gas in the QHE regime (Schweitzer *et al* 1984, 1985, Gudmundsson *et al* 1988). In this paper results will be presented for $P(E_\alpha)$ and $D_2(E_\alpha)$ on the tight binding model. These results will be compared with the results obtained for the continuum model (Aoki 1986, Ono *et al* 1989).

The tight binding Hamiltonian for the two-dimensional electron gas in the Landau gauge, $\boldsymbol{A} = (0, Bx, 0)$, is defined as (Schweitzer *et al* 1984)

$$(H\varphi)(x,y) = V(x,y)\varphi(x,y) + \varphi(x+1,y) + \varphi(x-1,y)$$
$$+ \exp(-\mathrm{i}2\pi xB)\varphi(x,y+1) + \exp(\mathrm{i}2\pi xB)\varphi(x,y-1) \tag{4}$$

where φ is some vector on the lattice Λ, and with $e = \hbar = m = 1$ the magnetic field $\boldsymbol{B}$ is expressed as the number of flux quanta per unit lattice cell. The lattice is chosen to be a square with unit lattice constant and L sites on each side. The potential $V(x, y)$ is taken to be a set of identically distributed independent random variables on the lattice sites. The uniform distribution is used with width W. Periodic boundary conditions are applied in both directions. Only magnetic fields which make the exponential factors commensurable with the lattice periodicity are considered. The dimension D_2 (see (3)) of the zero-disorder states is 2, and this is independent of the boundary conditions, which is not the case for the continuum model where D_2 is 1. The case of periodic boundary conditions is subtle because of the degeneracies but there is a symmetry breaking in the representation of the eigenstates which means that D_2 is 1 for each state in the representation (Grümm 1984). This is the first difference to be noticed between this model and the continuum model and it will be important in the discussion later.

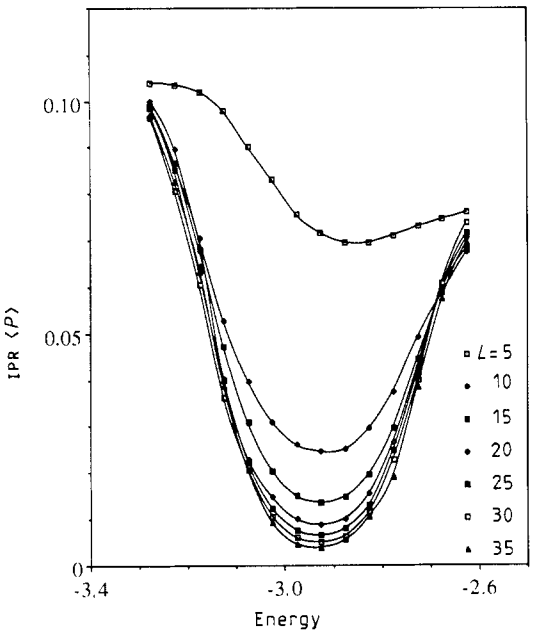

Figure 1. The inverse participation ratio averaged over intervals $\Delta E = 0.05$ is shown for the entire first magnetic subband. The magnetic field is $B = \frac{1}{5}$, the disorder strength $W = 1$ and lattices of size $L = 5, 10, 15, 20, 25, 30, 35$ have been used. The number of realisations for the disorder potential are 10000(5), 1000(10, 15, 20), 100(25), 50(30) and 10(35).

The IPR is obtained by diagonalising the Hamiltonian matrix numerically, finding the complete set of eigenvectors and constructing $P(E_\alpha)$ for each of them using (1). Shown in figure 1 is the IPR averaged over the N eigenvalues contained in an energy interval ΔE

$$\langle P \rangle_{\Delta E} = \sum_{E_\alpha \in \Delta E} \frac{P(E_\alpha)}{N} \tag{5}$$

calculated in the first subband for various lattice dimensions L, magnetic field $B = \frac{1}{5}$ and for disorder of strength $W = 1.0$. Notice that the energy of the smallest $\langle P \rangle_{\Delta E}$ changes with the size of the lattice but seems to stabilise for L greater than 20. The averaged IPR for energies near the centre of the band decrease with increasing L whereas away from the centre the IPR remains almost constant (lower energy side) or first decreases then increases again (higher energy side). It makes extrapolation difficult if the IPR scales non-monotonically with the length. Since the choice of the energy interval ΔE for averaging is arbitrary, only an arithmetic average of the most extended state of the first Landau band, taken over different realisations of the disorder potential, will be considered in what follows.

In figure 2 an attempt at extrapolation of the IPR P of the most extended state in the first subband is shown as a function of L^{-2} for various values of the disorder parameter W. Only for almost zero-disorder W do the extrapolations go unambiguously to zero, which indicates an extended state. However, for W larger than about 0.1, this extrapolation gives only localised states. It should be noted that $W \simeq 0.1$ is a weak disorder for which the broadening of the density of states is very small and the Landau bands

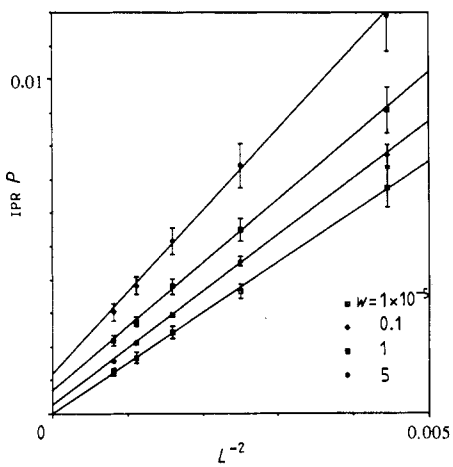

Figure 2. The IPR of the most extended states in the first subband plotted against $1/L^2$ for disorder parameters $W = 10^{-5}$, 0.1, 1 and 5. Once again, the magnetic field $B = \frac{1}{5}$. The number of realisations are 100(15, 20, 25, 30), 10(35), and the error bar is the standard deviation.

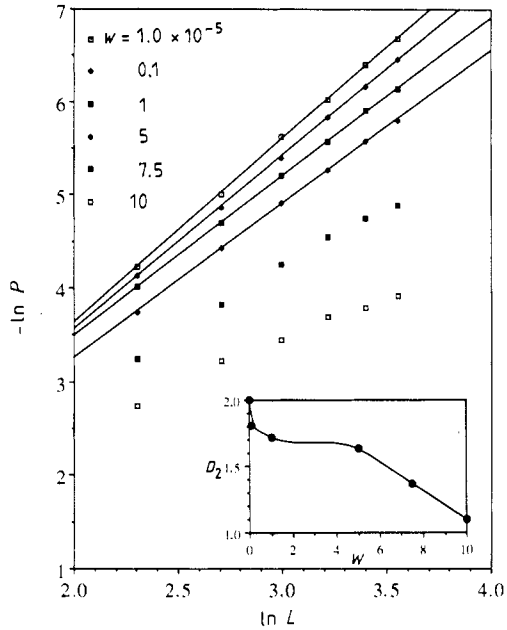

Figure 3. The logarithm of the most extended IPR in the first subband plotted against $\ln(L)$. The straight line at high L allow an estimation of D_2 to be made. $W = 10^{-5}$, 0.1, 1, 5, 7.5, 10 and $B = \frac{1}{5}$. The number of realisations are as in figure 2. In the inset an estimate of the Renyi dimension D_2 is shown as a function of the disorder strength W.

remain well separated. Unfortunately the equivalent data for the continuum model have not been published.

Another method of extracting asymptotic properties of the most extended states is to estimate the Renyi dimension $D_2(E_\alpha)$. Plotted in figure 3 is the negative of the logarithm of the smallest IPR P against the logarithm of L. From this plot D_2 can be estimated as a function of W. The results are shown in the inset. For disorder strengths $W = 7.5$ and $W = 10$ a good straight line for large L is not obtained. In this case the subbands already exhibit an appreciable overlap. It is probable that D_2 is much smaller than 1.0, even zero. However, for disorder strength $W < 7.5$ a saturation of P at system sizes larger than $L = 35$ can also not be excluded. A value of D_2 greater than zero indicates that the state is extended. Note that the results for the extrapolations obtained from figure 2 contradict those from figure 3. This means that great caution must be exercised in discussing these results because at least one extrapolation must be giving false results. The study of larger systems could in principle decide this question. Then, however, a better suited method for diagonalising the Hamiltonian matrix, perhaps a Lanczos algorithm, must be applied. It is possible that an asymptotically localised state may have stable fractal structure associated with a range of length scales, although D_2 is zero in the limit $L \to \infty$. Similar phenomena are known in other areas where the fractal concept is employed. For example, consider percolation on a lattice. A Hausdorff dimension can be associated with the percolating backbone but only on length scales orders of magnitude larger than the lattice constant.

As seen from a calculation of D_2 for the most extended state in the first subband, the main difference between the continuum and lattice models is that a small amount of disorder decreases D_2 in the tight binding model whereas it increases D_2 in the continuum model. For the lattice the rate of change of D_2 with disorder W is small in the range $1.0 < W < 5.0$, where $D_2 = 1.7 \pm 0.1$. This agrees well enough with calculations of D_2 for the continuum model (Aoki 1986, Kramer *et al* 1988, Ono *et al* 1989), but in the discussion they give there is no indication that the dependence of D_2 on the strength of the disorder has been considered. It is, however, clear from the results presented in this paper that there are important differences between the two models which means that universality cannot be invoked to permit the transfer of results from the continuum model to the lattice model. Recently, Ono *et al* (1989) observed the non-universality of D_2 even within the continuum model because D_2 is larger for potentials of finite range than for the δ-potential case.

The increase of D_2 with disorder for the state at the centre of the Landau band has been used to propose that for this reason the state can carry more current and so compensate for the localised states. However, this argument is not applicable to the tight binding model because D_2 decreases monotonically with increasing disorder. It is well known that one gets similar results for the Hall conductivity σ_{xy} calculated with the Kubo formula in the continuum model (Ando 1984) and in the lattice model (strip of finite width: Schweitzer *et al* 1985; finite systems with periodic boundary conditions in both directions: Johnston and Schweitzer (unpublished)). This implies that the increase found on D_2 with disorder in the continuum model is not essential for obtaining the quantum Hall effect within the localisation model.

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